

INTEGRABLE MANY-BODY SYSTEMS OF CALOGERO-MOSER-SUTHERLAND TYPE IN HIGH DIMENSION

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The many-body problems of Calogero-Moser-Sutherland type have a long history now. They are used in different areas of physics including conformal field theory and gauge theories. The detail review of the problem and references can be found in [1]. In recent papers [1,2] a treatment of elliptic Calogero-Moser-Sutherland system was suggested, which allows to obtain this system together with the complete set of its integrals starting up from the interacting Higgs field and vector field on the elliptic curve (the complete integrability of the system was obtained in [3] using inverse scattering technique). The method of [1,2] is the one suggested in [4,5], i.e. the method of Hamiltonian reduction. From the more general point of view [2] such type of constructions is connected with some moduli space of holomorphic G -bundles over smooth Riemann surfaces, where G is semisimple finite-dimensional Lie group.

We suggest in this paper a construction which is very close to the one of [1,2] but which is connected with Krichever-Novikov (KN) algebras of affine type which were introduced in [6] in connection with quantization of multiloop string diagrams and soliton theory.

We consider a system, which consists of a vector field and of a Higgs field with a free hamiltonian on a Riemann surface of nonzero genus. In mathematical words we consider an element of a KN-algebra and an element of its current subalgebra. Being subjected to the Hamiltonian reduction procedure that system provides a kind of many-body problem which turns out to be the problem of Calogero-Moser-Sutherland type. The dimension of its configuration space equals to the genus of the Riemann surface under consideration. One of the fields under consideration provides the configuration parameters of a certain many-body problem, while the invariants of the other field provide its integrals. It turns out (Theorem 2.1 below) that the number of configuration parameters is equal to the number of invariants, which proves the complete integrability of the obtained many-body problem in Liouville sense. In case of an elliptic curve we come to the result of [1,2] on the Calogero-Moser-Sutherland systems in elliptic case.

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1. KRICHEVER-NOVIKOV ALGEBRAS OF AFFINE TYPE

Let Γ be a compact algebraic curve over \mathbb{C} with two marked points P_{\pm} , \mathcal{A}^{Γ} be the algebra of meromorphic functions on Γ regular outside the points P_{\pm} , e be a

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vector field on Γ with the same analitical properties, \mathfrak{g} be a complex semisimple Lie algebra,

$$\mathcal{G} = \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{A}^{\Gamma} \oplus \mathbb{C}c \oplus \mathbb{C}\nabla \quad (1.1)$$

be a KN-algebra of affine type [6,7] with the one-dimensional centre generated by c , ∇ be the dirivative along the vector field e . The elements of the algebra \mathcal{G} will be denoted by $\tilde{X} = X + ac + be$, where $X \in \mathfrak{g} \otimes_{\mathbb{C}} \mathcal{A}^{\Gamma}$, $a, b \in \mathbb{C}$. \mathcal{G} is considered to be identified with its dual space \mathcal{G}^* by means the nondegenerate form

$$\langle X + ac + be, Y + a'c + b'e \rangle = \frac{1}{2\pi i} \oint_{c_0} (X, Y) \frac{dp}{E(p)} + ab' + ba', \quad (1.2)$$

where (\cdot, \cdot) is the Cartan-Killing form on \mathfrak{g} and in local coordinates $e = E(p)(dp)^{-1}$. A structure of the central extension on \mathcal{G} is given by means the cocycle

$$\gamma(X, Y) = \frac{1}{2\pi i} \oint_{c_0} (X, dY). \quad (1.3)$$

It should be mentioned that the analog of the relation (1.2) in [1] (see relation (4.1) there) necessarily has an arbitrariness in the choice of an integration 2-form, so it can not be generalized on the case $g > 1$ in an invariant way.

Adjoint action:

$$\begin{aligned} (\text{Ad}g)(X + ac + be) &= ac + be + gXg^{-1} - b(\nabla g)g^{-1} \\ &+ (\langle g^{-1}(\nabla g), X \rangle - \frac{b}{2} \langle (\nabla g)g^{-1}, (\nabla g)g^{-1} \rangle)c. \end{aligned} \quad (1.4)$$

In order to classify the *orbits* of adjoint action let us consider the monodromy equation

$$(\nabla + X)\Psi = 0 \quad (1.5)$$

and let Ψ_X be the multivalued solution of this equation satisfying the initial condition $\Psi_X(\gamma_0) = 1$ in some fixed point $\gamma_0 \in \Gamma$. Let G_X be the corresponding monodromy group.

Theorem 1.1 [7]. *Two elements $\nabla + X, \nabla + Y \in \mathcal{G}$ belong to the same orbit of adjoint action if and only if $\langle X, X \rangle = \langle Y, Y \rangle$ and there exists such a $g \in G$ that $gG_Xg^{-1} = G_Y$ (where $G = \exp \mathfrak{g}$).*

If G_X is an abelian group, the corresponding orbit is said to be *commutative*. In that case up to conjugation one has

$$\Psi_X = \exp \int_{\gamma_0}^{\gamma} \omega_X, \text{ where } \omega_X = -(d\Psi_X)\Psi_X^{-1} \in \mathfrak{h} \otimes \Omega_1(\Gamma) \quad (1.6)$$

and $\Omega_1(\Gamma)$ is the space of 1-forms on Γ which are meromorphic on Γ and regular out the points P_{\pm} . The relation (1.6) establishes the *duality* between the currents X and differentials ω_X . In other way the same duality can be established by means the nondegenerate form (1.2).

Lemma 1.1. *The duality between currents and differentials established by means the bilinear form (1.2) coincides with the duality established by the relation (1.6) and in local coordinates looks as follows:*

$$X \frac{dp}{E(p)} = \omega_X. \quad (1.7)$$

The *Weil group* W is defined as the group classifying commutative orbits. Let Q^\vee be the dual root lattice and \overline{W} be the Weil group of Lie algebra \mathfrak{g} , then $W = \overline{W}(Q^\vee)^{2g+1}$ [7]. The group W acts on the space $\mathfrak{h} \otimes \Omega_1(\Gamma)$ by means transforming the periods of differentials of that space.

From now on we shall suppose that Ω_1 is the $2g + 1$ -dimensional linear space which consists of such 1-forms on Γ that are holomorphic outside the points P_\pm , and have the orders not less than -1 and $-g$ in the points P_+ and P_- respectively.

Theorem 1.2 [7]. *The space of commutative orbits is isomorphic to the quotient $\Omega = \mathfrak{h} \otimes \Omega_1/W$.*

2. THE DYNAMICAL SYSTEM

A point of the *phase space* consists of the element $\nabla + X \in \mathcal{G}$ ("vector field" in physical terms), of a function $\phi : \Gamma \rightarrow \mathfrak{g}$ with an unique exponential singularity in P_- , and of a \mathfrak{g} -valued 1-form ω of the $(2g + 1) \times \dim \mathfrak{g}$ -dimensional space $\mathfrak{g} \otimes \Omega_1$. We shall suppose X to satisfy the relation $\text{res}_{P_+}(X \frac{dp}{E(p)}) = 0$.

The *gauge action* is as follows:

$$X \rightarrow gXg^{-1} - (\nabla g)g^{-1}, \quad \phi \rightarrow g\phi g^{-1}, \quad \omega \rightarrow g\omega g^{-1} - (dg)g^{-1}, \quad (2.1)$$

where g is a group current.

The *symplectic form*. Let $X \mapsto \omega_X = X \frac{dp}{E}$, A_k^X and B_k^X be the a-periods and the b-periods of ω_X respectively, A_k^ω and B_k^ω be the same for the 1-form ω ($k = 1, \dots, g$). The symplectic form S is defined by the relation

$$S = \frac{1}{2\pi i} \oint_{c_0} (\delta X \wedge \delta \phi) \frac{dp}{E} + \frac{1}{2} \sum_{j=1}^g (\delta A_j^X \wedge \delta B_j^\omega + \delta A_j^\omega \wedge \delta B_j^X), \quad (2.2)$$

where for any $\delta X, \delta Y \in \mathfrak{g}$ one has

$$\delta X \wedge \delta Y = \sum_{\alpha} \delta X_{\alpha} \wedge \delta Y_{-\alpha} + \sum_{k=1}^l \delta X_{\alpha_k} \wedge \delta Y_{\alpha_k}, \quad (2.3)$$

and

$$\begin{aligned} \delta X &= \sum_{\alpha} \delta X_{\alpha} e_{\alpha} + \sum_{k=1}^l \delta X_{\alpha_k} h_{\alpha_k}, \\ \delta Y &= \sum_{\alpha} \delta Y_{\alpha} e_{\alpha} + \sum_{k=1}^l \delta Y_{\alpha_k} h_{\alpha_k} \end{aligned}$$

are the canonical decompositions of the elements δX , δY respectively ($l = \text{rank } \mathfrak{g}$, α being the roots of \mathfrak{g} , α_k being the simple roots of \mathfrak{g} and h_{α_k} being their dual elements ($k = 1, \dots, l$)).

The *moment map*, which corresponds to (2.2), reads

$$\mu(X, \phi, \omega) = [\nabla + X, \phi] + \frac{1}{2} \sum_{j=1}^g [A_j^X, B_j^\omega] + \frac{1}{2} \sum_{j=1}^g [A_j^\omega, B_j^X]. \quad (2.4)$$

We put the *hamiltonian* of the system to be equal

$$H = \sum_{j=1}^g (A_j^\omega, A_j^\omega) + g \sum_{\alpha \in R_+} \phi_\alpha \phi_{-\alpha}, \quad (2.5)$$

where R is the root system of \mathfrak{g} and $\phi = \sum_{\alpha \in R} \phi_\alpha e_\alpha + \sum_{k=1}^l \phi_k h_{\alpha_k}$ is the canonical decomposition of the element $\phi \in \mathfrak{g}$.

Now we are in position to carry out a kind of *hamiltonian reduction* which results in the required many-body problem. First of all let us restrict our phase space by the two g -invariant conditions:

- 1⁰ $\omega_X = \omega$
- 2⁰ monodromy group G_X is abelian.

Remark 2.1. Denote $\Psi = \exp \int_{\gamma_0}^{\gamma} \omega$, then the conditions 1⁰, 2⁰ imply

$$(\nabla + X)\Psi = 0. \quad (2.6)$$

Now we want to reduce the system on the zero level of the moment map, i.e. on the manifold $\mu(X, \phi, \omega) = 0$. Because of the 2⁰ both sums in (2.4) vanish and *the equation of zero moment* reads

$$[\nabla + X, \phi] = 0 \quad (2.7)$$

Remark 2.2. Both equations (2.6) and (2.7) appear in the Krichever's algebraic-geometrical construction of the solutions of zero curvature equations [10].

The equation (2.7) can be rewritten in the form

$$\nabla \phi + (\text{ad} X)\phi = 0. \quad (2.8)$$

For $\phi = \sum_{\alpha \in R} \phi_\alpha e_\alpha + \sum_{k=1}^l \phi_k h_{\alpha_k}$ one has $(\text{ad} X)\phi = \sum_{\alpha \in R} \alpha(X)\phi_\alpha$. Furthermore $\alpha(X)$ satisfies the relation $\alpha(X \frac{dp}{E}) = \omega_X(h_\alpha, \cdot)$ $h_\alpha \in \mathfrak{h}$ (h_α is uniquely defined). So the vector equation (2.7) (which appears as the matrix equation in case of classical Lie algebras \mathfrak{g}) is equal to the following system of scalar equations:

$$d\phi_\alpha + \omega_X(h_\alpha)\phi_\alpha = 0 \quad (\alpha \in R, \alpha \neq 0), \quad (2.9)$$

$$d\phi_\alpha = 0 \quad (\alpha \in R, \alpha = 0). \quad (2.10)$$

The 1-form $\omega_X(h_\alpha)$ is \mathbb{C} -valued, so the equation (2.9) has the unique up to the scalar factor solution, namely the Baker-Achieser function

$$\phi_\alpha(\gamma) = \left(\exp \int_{\gamma_0}^{\gamma} \omega_X(h_\alpha) \right) \frac{\theta(A(\gamma) + Z(D) + U_\alpha)}{\theta(A(\gamma) + Z(D))\theta(Z(D) + U_\alpha)}, \quad (2.11)$$

where $\gamma, \gamma_0 \in \Gamma$, and $\gamma_0 \neq P_\pm$ is fixed, θ is the Riemann theta-function, A is the transformation of Abel, $Z(D) = -A(D) - \mathcal{K}$ (\mathcal{K} being the the vector of Riemann constants), and U_α is the vector of b-periods of the differential $\omega_X(h_\alpha)$. Note that the factor $\theta((Z(D) + U_\alpha))$ in the denominator of (2.11) is simply a constant. The reason of the choice of this constant will be clear at once.

Remark 2.3. It will be shown in Section 3 that the factor $\theta(Z(D) + U_\alpha)$ is also related to the Weil-invariance of the hamiltonian.

Lemma 2.1. *If $\gamma = \gamma_0$ and $\alpha \neq 0$ then $\phi_\alpha(\gamma_0) = \frac{1}{\theta(Z(D))}$, and $\sum_{\alpha \in R} \phi_\alpha(\gamma_0)e_\alpha = \frac{1}{\theta(Z(D))}J$, where $J = \sum_{\beta \in R_+} (e_\beta + e_{-\beta})$.*

Proof. If $\gamma = \gamma_0$ then $\exp \int_{\gamma_0}^{\gamma} \omega_X(h_\alpha) = 1$, $A(\gamma) = 0$ and so $\phi_\alpha(\gamma_0) = \frac{\theta(Z(D) + U_\alpha)}{\theta(Z(D))\theta(Z(D) + U_\alpha)} = \frac{1}{\theta(Z(D))}$ independently of α .

Example 2.1. Let $\mathfrak{g} = \mathfrak{gl}(N)$. Then

$$J = \begin{pmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 0 \end{pmatrix}.$$

It remains only to resolve the equation (2.10) in order to complete the reduction. One has from (2.10) $\phi_{\mathfrak{h}} = \text{const}$ ($k = 1, \dots, l$), where $\phi_{\mathfrak{h}} = \sum_k \phi_k h_{\alpha_k}$. Let us correspond to j -th handle of Γ its own constant $\phi_{\mathfrak{h}}^{(j)}$ and respectively its own solution $\phi^{(j)}$ of (2.9)-(2.10), which satisfy the condition

$$\frac{1}{\theta(Z(D))}J + A_j^\omega = \phi^{(j)}(\gamma_0). \quad (2.12)$$

in any gauge where $X \in \mathfrak{h}$. This relation does not depend on the choice of such kind a gauge. In fact A_j^ω and $\phi^{(j)}(\gamma_0)$ are transformed by the low $(\cdot) \mapsto g_0(\cdot)g_0^{-1}$, where $g_0 = g(\gamma_0)$. If $(\text{Ad } g_0)\mathfrak{h} = \mathfrak{h}$ then the action of g_0 reduces to the action of an element of the Weil group of \mathfrak{g} , but J is Weil-invariant.

As in the gauge where $X \in \mathfrak{h}$ noncartanian part of $\phi(\gamma_0)$ is *constant* and is equal to $J/\theta(Z(D))$ (Lemma 2.1), all the conditions (2.12) are compatible and have the solution

$$A_j^\omega = \phi_{\mathfrak{h}}^{(j)} \quad (j = 1, \dots, g). \quad (2.13)$$

Lemma 2.2. *The integral term on the right hand in (2.2) vanishes on the reduced phase space.*

Proof. In the gauge where $\delta X \in \mathfrak{h}$ one has $\delta X \wedge \delta \phi = \delta X \wedge \delta \phi_{\mathfrak{h}}$ (see (2.3)). As it was just now shown, $\phi_{\mathfrak{h}}$ is a constant in γ . Furthermore

$$\frac{1}{2\pi i} \oint_{c_0} (\delta X \wedge \delta \phi_{\mathfrak{h}}) \frac{dp}{E} = \frac{1}{2\pi i} \left(\oint_{c_0} \delta X \frac{dp}{E} \right) \wedge \delta \phi_{\mathfrak{h}} = \delta \left(\frac{1}{2\pi i} \oint_{c_0} X \frac{dp}{E} \right) \wedge \delta \phi_{\mathfrak{h}}.$$

There is nothing but $\delta(\text{res}_{P_+} X \frac{dp}{E}) \wedge \delta \phi_{\mathfrak{h}}$ on the right hand of the last relation. But $\text{res}_{P_+} X \frac{dp}{E} = 0$, therefore its variation vanishes.

In order to summarize the obtained results let us introduce a new notation. Denote

$$P_j = A_j^X = A_j^\omega, \quad Q_j = B_j^X = B_j^\omega \quad (j = 1, \dots, g) \quad (2.14)$$

and define the vectors $U_k \in \mathbb{C}^g$ ($k = 1, \dots, l$) from the relation

$$(Q_1, \dots, Q_g) = (U_1, \dots, U_l)^T. \quad (2.15)$$

U_k will be considered as the vector of the b-periods of the differential $\omega(h_{\alpha_k})$ ($k = 1, \dots, l$), where $\alpha_1, \dots, \alpha_l$ are the simple roots of \mathfrak{g} . For an arbitrary root α the vector U_α and U_1, \dots, U_l are related as follows: if $\alpha = \sum_{k=1}^l m_k \alpha_k$ (m_k are nonnegative integers), then

$$U_\alpha = \sum_{k=1}^l m_k U_k. \quad (2.16)$$

The procedure of hamiltonian reduction, which has been carried out in this Section, results in the following dynamical system, which will be called *the reduced system*. The phase parameters of the reduced system are as follows: $P_j, Q_j \in \mathfrak{h}$ ($j = 1, \dots, g$). The symplectic structure is given by the standard relation

$$S_{red} = \sum_{j=1}^g P_j \wedge Q_j \quad (2.17)$$

(it follows from (2.2) because of (2.14) and Lemma 2.2). Finally the hamiltonian of the reduced system is of the form

$$H = \sum_{j=1}^g \sum_{k=1}^l \phi_{jk}^2 + g \sum_{\alpha > 0} \phi_\alpha \phi_{-\alpha}, \quad (2.18)$$

where ϕ_α is given by (2.11), U_α are related to the phase parameters by (2.15) and (2.16) and ϕ_{jk} are defined by the relation $\phi_{\mathfrak{h}}^{(j)} = \sum_{k=1}^l \phi_{jk} h_{\alpha_k}$ ($j = 1, \dots, g$).

Example 2.2. Let $\mathfrak{g} = \mathfrak{gl}(N)$. Then $l = N$, an arbitrary root is of the form $\alpha = \alpha_{ik}(h) = h_i - h_k$ ($h = \text{diag}(h_1, \dots, h_N)$), $\omega(h_{\alpha_{ik}}) = \omega_i - \omega_k$ ($\omega = \text{diag}(\omega_1, \dots, \omega_N)$) and therefore $U_\alpha = U_i - U_k$. Therefore (2.18) reads

$$H = \sum_{j=1}^g (P_j, P_j) + g \sum_{i < k} \frac{\theta(A(\gamma) + Z(D) + U_i - U_k) \theta(A(\gamma) + Z(D) + U_k - U_i)}{\theta(A(\gamma) + Z(D))^2 \theta(Z(D) + U_i - U_k) \theta(Z(D) + U_k - U_i)}, \quad (2.19)$$

where U_i ($i = 1, \dots, N$) are related with Q_j ($j = 1, \dots, g$) via (2.15), (2.16). It is evident, that (2.19) is the many-body problem in g -dimensional space with the pairwise interaction.

Theorem 2.1. *The reduced hamiltonian system with the hamiltonian (2.18) and the symplectic structure (2.17) is completely integrable in sense of the Liouville theorem.*

Proof. By definition $P_j, Q_j \in \mathfrak{h}$ ($j = 1, \dots, g$) and $\dim \mathfrak{h} = l$. Therefore the dimension of the phase space is equal to $2gl$.

On the other hand for each $j = 1, \dots, g$ the Chevalley invariants of the \mathfrak{g} -valued function $\phi^{(j)}$ appear as the integrals of the system. For each j the function $\phi^{(j)}$ has $\text{rank } \mathfrak{g} = l$ invariants, so the system has gl independent integrals. Applying instead the basic symmetric functions of g variables to each set of the same Chevalley invariants differing by j only one can obtain the family of gl independent integrals in involution. The Theorem is proved.

In conclusion let us consider the example 2.2 with $g = 1$ in more detail.

Example 2.3. Let $\mathfrak{g} = \mathfrak{gl}(N)$, $g = 1$ (Γ is an elliptic curve). Then by Abel transformation Γ can be identified with the torus \mathbb{T}^2 . In other words one can denote $A(\Gamma) = z \in \mathbb{T}^2$. As $g = 1$ then by choice of the initial point γ_0 the constant $Z(D)$ can be vanished. Thus in case $g = 1$

$$H = (P_1, P_1) + \sum_{i < k} \frac{\theta(z + U_i - U_k) \theta(z - U_i + U_k)}{\theta(z)^2 \theta(U_i - U_k)^2}. \quad (2.20)$$

Because of the addition theorem

$$\frac{\theta(z + u) \theta(z - u)}{\theta(z)^2 \theta(u)^2} = \mathfrak{P}(u) - \mathfrak{P}(z),$$

where \mathfrak{P} is the Weierstrass \mathfrak{P} -function. Therefore

$$H = (P_1, P_1) + \sum_{i < k} (\mathfrak{P}(U_i - U_k) - \mathfrak{P}(z)), \quad (2.21)$$

that coincides with *elliptic Calogero–Moser–Sutherland system* (cf. [1, 2]).

Remark 2.4. The reduced symplectic structure (2.17) is exactly a kind of the symplectic structure due to Howe, which is extensively exploited in the number theory investigations [8].

3. WEIL-INVARIANCE OF THE REDUCED HAMILTONIAN

The phase space of the reduced dynamical system (2.17)-(2.18) is endowed with the natural W -action, where W is the Weil group of \mathcal{G} (see Section 1). Now we want to return to the Remark 2.3 and to show that the choice of the scalar factor $\theta(Z(D) + U_\alpha)$ in (2.11) is related to the Weil-invariance of the hamiltonian.

Theorem 3.1. *The hamiltonian H , defined by (2.18), is W -invariant.*

Proof. Let \overline{W} be the Weil group of the Lie algebra \mathfrak{g} , L be the lattice of periods of the Riemann surface Γ . As it was above mentioned (Sect. 1) the action of W on the space $\mathfrak{h} \otimes \Omega_1(\Gamma)$ results, first, in the translations of periods of 1-forms by elements of L , and, second, in the action of \overline{W} on \mathfrak{h} -cofactor. Each of the summands in (2.18) is evidently \overline{W} -invariant, so it suffices to prove its L -invariance.

Let $\omega \in \mathfrak{h} \otimes \Omega_1(\Gamma)$. Translating the periods of ω by the elements of L is equivalent to the shift $\omega(h_\alpha) \mapsto \omega(h_\alpha) + \omega_{int}$, where $\omega_{int} = \sum_{k=1}^g n_k \omega_k$, ω_k are the basic holomorphic differentials ($k = 1, \dots, g$). It remains to prove the invariance of ϕ_α in respect to such kind a shift and, moreover, it suffices to take $\omega_{int} = \omega_k$ ($k = 1, \dots, g$).

Note that $\exp \int_{\gamma_0}^\gamma \omega_k = \exp(2\pi i A_k(\gamma))$ (A_k is the k -th component of the Abel transformation). Moreover, under the above mentioned shift $U_\alpha \mapsto U_\alpha + \frac{1}{2\pi i} \oint_b \omega_k = U_\alpha + B_k$ (where B_k is the k -th column of the matrix of b-periods). The latest mapping results in the multiplying the $\theta(A(\gamma) + Z(D) + U_\alpha)$ by $\exp(-\pi i(B_{kk} + 2A_k(\gamma) + Z_k(D) + (\overline{U}_\alpha)_k))$ in (2.11) and in the multiplying the $\theta(Z(D) + U_\alpha)$ by $\exp(-\pi i(B_{kk} + 2(U_\alpha)_k + Z_k(D)))$ (index k denotes the k -th coordinate of a vector). Thus the ratio of θ 's in (2.11) gets the factor $\exp(-2\pi i A_k(\gamma))$, that compensates the variation of the $\exp \int_{\gamma_0}^\gamma \omega(h_\alpha)$. Thus ϕ_α is L -invariant for each α , and because for (2.18) H is gauge invariant.

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